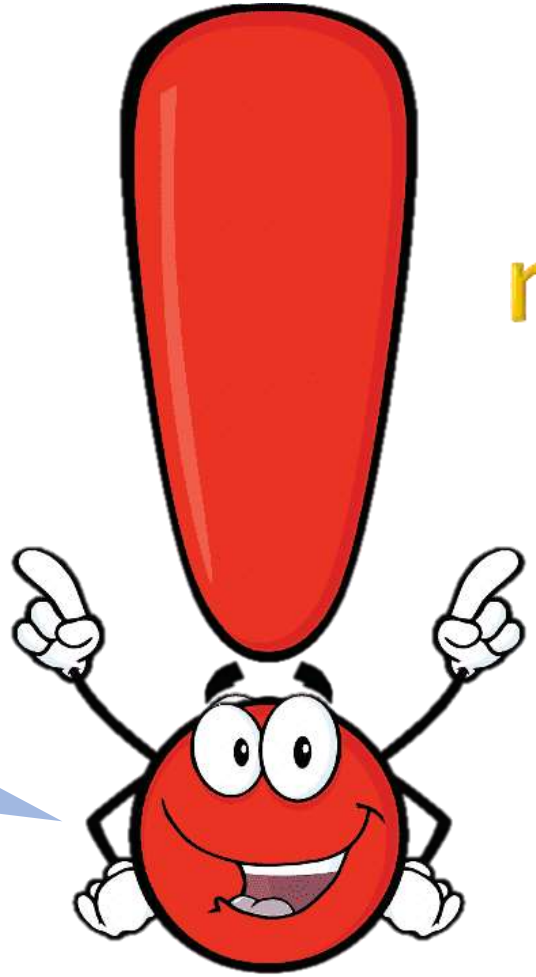
Permutations

Factorial notation

n is a natural number
Factorial of a number n is:

n!

Not just for language anymore!



$$n! = n \times (n-1) \times \ldots \times 2 \times 1$$

Example:

$$2! = 2 \times 1 = 2$$
$$3! = 3 \times 2 \times 1 = 6$$
$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Factorial notation

Properties

✓ By convention $0! = 1$

✓ $n! = n \times (n-1)!$

$(n-1)! = (n-1)\times(n-2)\times\ldots\times2\times1$

$n! = n \times (n-1) \times\ldots \times2 \times1$

$(n-1)!$

Example:

$6! = 6\times5!$

$10! = 10\times9!$

✓ $(kn)! \neq kn!$

$(3\times2)! = 6! = 6\times5 \times4 \times3 \times2 \times1=720$

$3 \times2! = 3 \times2 \times1 = 6 \neq 720$

✓ $(k + n)! \neq k! + n!$

✓ $(k\times n)! \neq k! \times n!$

$(3\times2)! = 6! = 6\times5 \times4 \times3 \times2 \times1=720$

$3! \times2! = 3 \times2 \times1\times2\times1 = 12 \neq 720$

$(3+2)! = 5! = 120$

$3!+2! = 6+2 = 8 \neq 120$

Permutation

Definition

A permutation is an arrangement of all or part of a set of objects, with regard to the ***order*** of the arrangement.

Example:
Given the set of 3 letters {A,B,C}.
How many ways can we arrange 2 letters from that set?

A – B – C

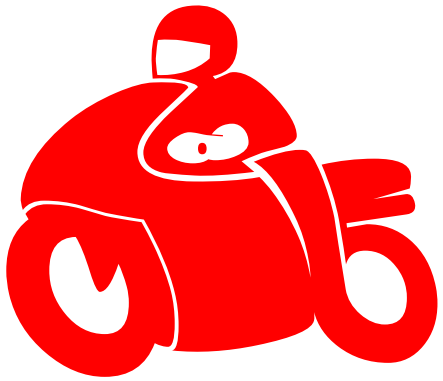
AB AC BA BC CA CB

6 permutations

Permutation

Types of permutation

1. Without repetition



First 3 places in a race

2. With repetition

The code of a lock

But how to calculate the number of permutations in each case???

Permutation

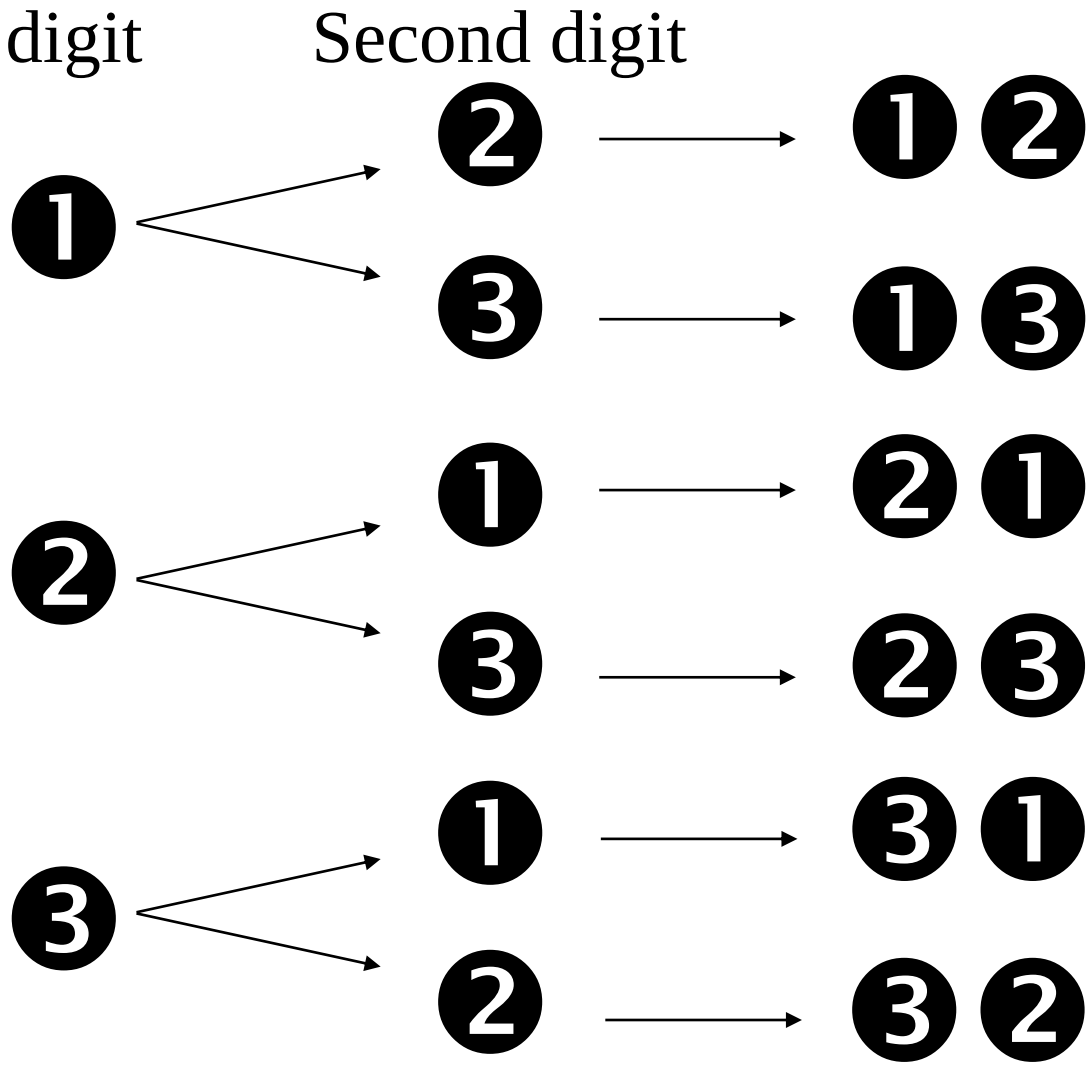
Types of permutation

1 Without repetition

❶ ❷ ❸

How many numbers of 2 distinct digits can be formed from the above list of numbers?

$3 \times 2 = 6$



6 possibilities

Number of possibilities of the first digit × Number of possibilities of the second digit

Permutation

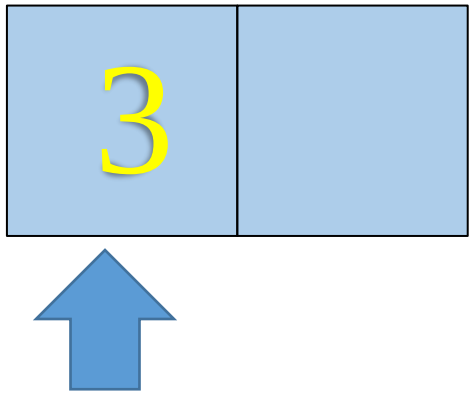
Types of permutation

How many numbers of 2 distinct digits can be formed from the above list of numbers?

$3 \times 2 = 6$

Number of possibilities of the first digit

Number of possibilities of the second digit



{1;2;3}

Permutation

Types of permutation

3	2

$= 3 \times 2 = 6$

1 Without repetition

❶ ❷ ❸

{1;2}
or {1;3}
or {2;3}

How many numbers of 2 distinct digits can be formed from the above list of numbers?

$3 \times 2 = 6$

Number of possibilities of the first digit × Number of possibilities of the second digit

Permutation

Types of permutation

1 Without repetition

❶ ❷ ❸

How many numbers of 2 distinct digits can be formed from the above list of numbers?

$3 \times 2 = 6$

Second method:

You can use the following formula:

$$A_n^r = \frac{n!}{(n-r)!}$$

In this case: $n = 3$; $r = 2$

$A_3^2 = \frac{3!}{1!} = \frac{6}{1} = 6$ possibilities.

Remark: the notation nPr is same as A_n^r

Permutation

Types of permutation

1 Without repetition

Application

❶ ❷ ❸

❹ ❺ ❻

How many permutations can be formed:

a) **of 4 distinct digits?**

b) **of 6 distinct digits?**

Permutation

Types of permutation

1 Without repetition

Application

❶ ❷ ❸

❹ ❺ ❻

How many permutations can be formed:

a) of 4 distinct digits?

b) of 6 distinct digits?

a) Permutation of 4 digits out of 6 digits, without repetition so: A_6^4

$= \frac{6!}{(6-4)!} = \frac{720}{2} = 360$

Or

6	5	4	3

$=6\times5\times4\times3=360$

Permutation

Types of permutation

1 Without repetition

Application

❶ ❷ ❸

❹ ❺ ❻

How many permutations can be formed:

a) of 4 distinct digits?

b) of 6 distinct digits?

b) Permutation of 6 digits out of 6 digits, without repetition so: $A_6^6 = \frac{6!}{(6-6)!} = \frac{720}{1}$

$= 720$

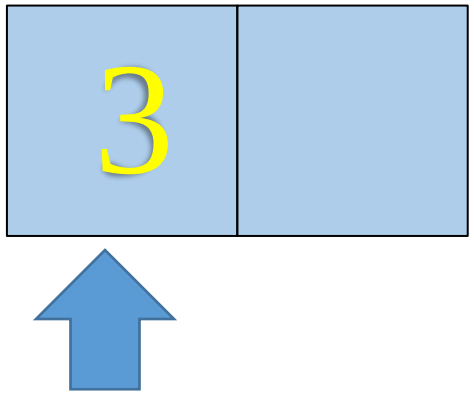
Or

6	5	4	3	2	1

$=6\times5\times4\times3\times2\times1=720$

Permutation

Types of permutation



2 With repetition

{1;2;3}

❶ ❷ ❸

How many numbers of 2 digits can be formed from the above list of numbers?

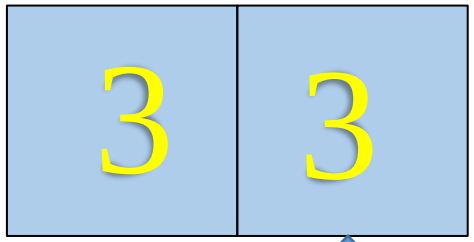
Number of possibilities of the first digit × Number of possibilities of the second digit

Permutation

Types of permutation

❶ ❷ ❸

How many numbers of 2 digits can be formed from the above list of numbers?



$= 3 \times 3 = 9$

{1;2;3}

Number of possibilities of the first digit

×

Number of possibilities of the second digit

Permutation

Types of permutation

2 With repetition

❶ ❷ ❸

How many numbers of 2 digits can be formed from the above list of numbers?

Permutation with repetition, with order is called r-lists : the number of possibilities is n^r

In this example: n = 3 and r = 2

So 2-lists of $3^2 = 9$ possibilities

Number of possibilities of the first digit × Number of possibilities of the second digit

Permutation

Types of permutation

2 With repetition

Application

❶ ❷ ❸

❹ ❺ ❻

How many permutations can be formed:

a) of 4 digits?

b) of 6 digits?

Permutation

Types of permutation

2 With repetition

Application

❶ ❷ ❸

❹ ❺ ❻

How many permutations can be formed:

a) of 4 digits?

b) of 6 digits?

a) Permutation of 4 digits out of 6 with repetition: 4-lists

The number of permutations is: $6^4 = 1296$

Or

6	6	6	6	=6×6×6×6=1296

Permutation

Types of permutation

2 With repetition

Application

❶ ❷ ❸

❹ ❺ ❻

How many permutations can be formed:

a) of 4 digits?

b) of 6 digits?

b) Permutation of 6 digits out of 6 with repetition: 6-lists

The number of permutations is:

$$6^6 = 45656$$

Or

6	6	6	6	6	6

$=6\times6\times6\times6\times6\times6=45656$

Permutation

Summary

- Permutation is an arrangement with regard to the order.
- Number of permutation *without repetition of* n elements is: n!
- Number of permutation *without repetition of* r *elements out of* n elements (r < n) is: $nPr = A_n^r = \frac{n!}{(n-r)!}$
- Number of permutation *with repetition of* r *elements out of* n *elements* $(r \leq n)$ is: n^r (r-lists)

Permutation

Applications

❶ In a race of 10 runners, how many ways can the first, the second and the third runners arrive?

3 places: r = 3

10 runners : n = 10

Permutation without repetition: $P_n^r = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 720$ possibilities

10	9	8

$= 10 \times 9 \times 8 = 720$

Permutation

Applications

❷ How many outcomes are there upon tossing a coin 3 times?

3 times: r = 3

Each time, there are 2 possibilities: Head H or Tail T : n = 2

Permutation with repetition: $n^r = 2^3 = 8$ possibilities

2	2	2

$= 2 \times 2 \times 2 = 8$

Permutation

Applications

❸ 3 boys and 4 girls are to be seated on a bench next to each other in a row.

a) In how many ways can they sit?

7 places and no repetition

Permutation of 7 persons is 7! = 5040 ways

Permutation

Applications

❸ 3 boys and 4 girls are to be seated on a bench next to each other in a row.

b) Calculate the number of ways they can be seated knowing that:

1. The boys are next to each other.

B	B	B	G	G	G	G	=3×2×1×4×3×2×1=144
G	B	B	B	G	G	G	=4×3×2×1×3×2×1=144
G	G	B	B	B	G	G	=4×3×3×2×1×2×1=144
G	G	G	B	B	B	G	=4×3×2×3×2×1×1=144
G	G	G	G	B	B	B	=4×3×2×1×3×2×1=144

144×5=720

Permutation

Applications

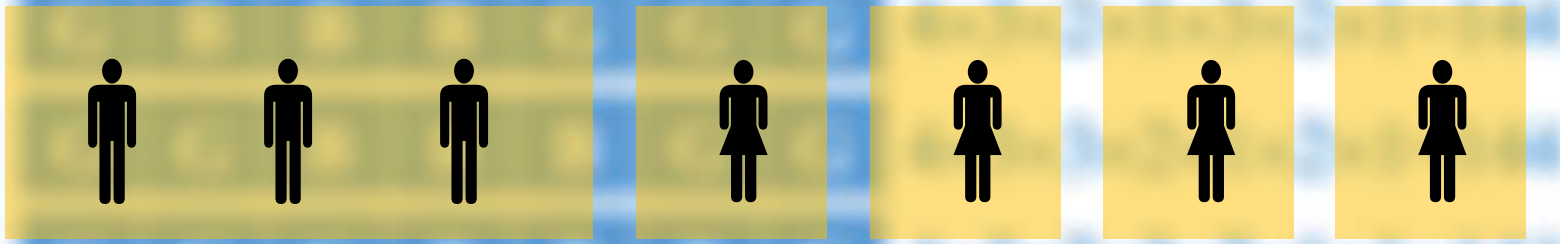
❸ 3 boys and 4 girls are to be seated on a bench next to each other in a row.

b) Calculate the number of ways they can be seated knowing that:

1. The boys are next to each other.

Suppose that the 3 boys are 1 group:

So permutation of 1+4=5



The number of permutations of 5 elements is 5!=120

The number of permutations of the boys in the group is 3!=6

So the total number of permutations is 120×6=720 possibilities

Permutation

Applications

❸ 3 boys and 4 girls are to be seated on a bench next to each other in a row.

b) Calculate the number of ways they can be seated knowing that:

2. No two of the same gender are beside each other.

G	B	G	B	G	B	G

$=3\times2\times1\times4\times3\times2\times1=144$

Permutation

Applications

❹ Given the digits: 0 – 1 – 2 – 3 – 4 – 5 – 6

1) How many 4 digit numbers can be formed?

Permutation with repetition of 4 out of 7 :

6	7	7	7

$= 6 \times 7 \times 7 \times 7 = 2058$ numbers

$\neq 0$

Permutation

Applications

❹ Given the digits: 0 – 1 – 2 – 3 – 4 – 5 – 6

2) How many 4 distinct digits can be formed?

Permutation with no repetition of 4 out of 7

But first digit must be ≠ 0 so :

6	6	5	4

= 6 ×6×5×4=720 numbers

Second method:

There is condition on the first digit so :

first digit: 1 out of 6 (1 – 2 – 3 – 4 – 5 – 6)

The other digits: 3 out of 6 (0 and the 5 remaining digits out of the other 6)

$A_6^1 \times A_6^3 = 6 \times 120 = 720$

Permutation

Applications

❹ Given the digits: 0 – 1 – 2 – 3 – 4 – 5 – 6

3) How many 3 distinct digit numbers can be formed such that the first digit is even and the last digit is odd?

Permutation without repetition of 3 out of 7 but with conditions:

3	5	3

$= 3 \times 5 \times 3 = 45$ numbers

$\neq 0$ $\{1;3;5\}$

$\{2;4;6\}$

Second method:

First digit: 1 out of 3 so A_3^1

Third digit: 1 out of 3 so A_3^1

Second digit: 1 out of the 5 remaining digits: A_5^1

So the number of possibilities is: $A_3^1 \times A_5^1 \times A_3^1 = 3 \times 5 \times 3 = 45$ numbers

Permutation

Applications

❺ In how many ways can the letters of the given words be arranged?

1) math

$$\frac{4!}{1!\times1!\times1!\times1!} = 12$$

2) facebook

$$\frac{8!}{1!\times1!\times1!\times1!\times1!\times2!\times1!} = 20160$$

3) Mississippi

$$\frac{11!}{1!\times4!\times4!\times2!} = 34650$$

Number of ways of arranging n letters of a word is:

$$\frac{n!}{\text{repetition}}$$